

THE PIECEWISE POLYNOMIAL COLLOCATION METHOD FOR NONLINEAR WEAKLY SINGULAR VOLTERRA EQUATIONS

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ABSTRACT. Second-kind Volterra integral equations with weakly singular kernels typically have solutions which are nonsmooth near the initial point of the interval of integration. Using an adaptation of the analysis originally developed for nonlinear weakly singular Fredholm integral equations, we present a complete discussion of the optimal (global and local) order of convergence of piecewise polynomial collocation methods on graded grids for nonlinear Volterra integral equations with algebraic or logarithmic singularities in their kernels.

1. INTRODUCTION

The solution of a second-kind Fredholm integral equation with weakly singular kernel is typically nonsmooth near the boundary of the domain of integration (its derivatives are unbounded). We refer to Richter [28], Pedas [21], [22], Schneider [30], Pitkäranta [24], [25], Vainikko and Pedas [41], Graham [8], Vainikko, Pedas and Uba [42], Vainikko [39], [40], Uba [34], Kaneko, Noren and Xu [12], R. Kangro [14], U. Kangro [15], [16], Pedas and Vainikko [23].

If one wants to obtain a high-order convergence in a numerical method for these equations one has to take into account, in some way, the singular behaviour of the exact solution. This can be done by using polynomial splines on special graded grids. The theory of graded grids in the approximation by polynomial splines goes back to Rice [27] (for a complete theory see, for example, de Boor [1] or Schumaker [32]). In the numerical solution of second-kind Fredholm integral equations with weakly singular kernels, graded grids were used by Chandler [7], Schneider [31], Vainikko and Uba [43], Graham [9], Vainikko, Pedas and Uba [42], Vainikko [37], [38], [40], Hackbusch [10], Uba [35], [36], Kaneko, Noren and Xu [13], Tamme [33], Pedas and Vainikko [24].

A similar situation arises in the case of Volterra integral equations. For second-kind Volterra equations with weakly singular kernels the nonsmooth behaviour of solutions occurs near the initial point of integration. We refer to Miller and Feldstein [20], de Hoog and Weiss [11], Logan [18], Lubich [19], Brunner [2], [4], Brunner and van der Houwen [6]. Again, if one is interested in finding an approximate solution which exhibits high-order accuracy, one may resort to approximation with polynomial splines on graded grids which reflect the singular behaviour of the exact

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solution near the initial point. In the numerical solution of second-kind Volterra integral equations with weakly singular kernels graded grids were used by Brunner [4], [3], Brunner and van der Houwen [6], and Brunner [5].

In the previous papers the case of Fredholm equations was considered independently of Volterra equations and, vice versa, the results for Volterra equations were obtained independently of existing results on Fredholm equations.

The purpose of the present paper is to show how in the analysis of the numerical solution of weakly singular Volterra integral equations by polynomial splines on graded grids it is possible to use corresponding existing results for Fredholm equations. More precisely, on the basis of results from [40, pp. 22–23] we shall study the smoothness of solutions and the piecewise polynomial collocation method for a rather wide class of nonlinear weakly singular Volterra integral equations. Using special collocation points in conjunction with graded grids, we derive global convergence estimates and analyze a superconvergence effect at the collocation points. The main results of the paper considerably extend known ones and are formulated in Theorems 2.1–2.4 (see Section 2). The proofs of these assertions are given in Section 4 and are based on the following simple idea: for a given Volterra equation

$$(1.1) \quad u(t) = \int_0^t K(t, s, u(s))ds + f(t), \quad 0 \leq t \leq T,$$

we find the appropriate extensions $\tilde{K}(t, s, u)$ and $\tilde{f}(t)$ for $K(t, s, u)$ and $f(t)$ so that from the corresponding results of Section 3 for the Fredholm equation

$$u(t) = \int_0^{2T} \tilde{K}(t, s, u(s))ds + \tilde{f}(t), \quad 0 \leq t \leq 2T,$$

we can derive all results formulated in Theorems 2.1–2.4 for the Volterra equation (1.1). Finally, in Section 5 we present some numerical illustrations.

2. VOLTERRA EQUATIONS WITH WEAKLY SINGULAR KERNELS

2.1. Smoothness of the solution. Consider the nonlinear Volterra equation

$$(2.1) \quad u(t) = \int_0^t K(t, s, u(s))ds + f(t), \quad 0 \leq t \leq T.$$

The following assumptions (V1)–(V3) are made.

(V1) The kernel $K = K(t, s, u)$ is m times ($m \geq 1$) continuously differentiable with respect to t , s , u for $t \in [0, T]$, $s \in [0, t)$, $u \in \mathbb{R}$, and there exists a real number $\nu \in (-\infty, 1)$ such that for $0 \leq s < t \leq T$, $u \in \mathbb{R}$, and for nonnegative integers i, j, k with $i + j + k \leq m$, the following inequalities hold:

$$(2.2) \quad \left| \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j \left(\frac{\partial}{\partial u} \right)^k K(t, s, u) \right| \leq b_1(|u|) \begin{cases} 1 & \text{if } \nu + i < 0, \\ 1 + |\log |t - s|| & \text{if } \nu + i = 0, \\ |t - s|^{-\nu - i} & \text{if } \nu + i > 0, \end{cases}$$

and

$$(2.3) \quad \left| \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j \left(\frac{\partial}{\partial u} \right)^k K(t, s, u_1) - \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j \left(\frac{\partial}{\partial u} \right)^k K(t, s, u_2) \right|$$

$$\leq b_2(\max\{|u_1|, |u_2|\})|u_1 - u_2| \begin{cases} 1 & \text{if } \nu + i < 0, \\ 1 + |\log |t - s|| & \text{if } \nu + i = 0, \\ |t - s|^{-\nu - i} & \text{if } \nu + i > 0, \end{cases}$$

where the functions $b_1 : [0, \infty) \rightarrow [0, \infty)$ and $b_2 : [0, \infty) \rightarrow [0, \infty)$ are assumed to be monotonically increasing.

(V2) $f \in C^{m,\nu}(0, T]$, i.e. $f(t)$ is m times continuously differentiable for $0 < t \leq T$ and the estimate

$$(2.4) \quad \left| f^{(k)}(t) \right| \leq \text{const}_f \begin{cases} 1 & \text{if } k < 1 - \nu, \\ 1 + |\log t| & \text{if } k = 1 - \nu, \\ t^{1-\nu-k} & \text{if } k > 1 - \nu, \end{cases} \quad t \in (0, T],$$

holds for $k = 0, 1, \dots, m$.

(V3) The integral equation (1.1) has a solution $u_0 \in L^\infty(0, T)$.

Notice that conditions (V1) and (V2) guarantee the existence and uniqueness of the solution to (2.1) on some interval $[0, T_0]$, $T_0 \leq T$. On $[0, T]$ the existence and uniqueness of the solution will be guaranteed if we impose the following global Lipschitz condition on $K(t, s, u)$: for $0 \leq t \leq T$, $0 \leq s < t$, $u_1, u_2 \in \mathbb{R}$, there holds

$$|K(t, s, u_1) - K(t, s, u_2)| \leq b|u_1 - u_2| \begin{cases} 1 & \text{if } \nu < 0, \\ 1 + |\log |t - s|| & \text{if } \nu = 0, \\ |t - s|^{-\nu} & \text{if } \nu > 0, \end{cases}$$

with a constant b independent of u_1 and u_2 (cf. (2.3), $i = j = k = 0$). We do not restrict the problem by this global Lipschitz condition, but assume (V3). Moreover, assuming (V3) we in fact could replace (V1) by the corresponding local condition for $t \in [0, T]$, $s \in [0, t)$, $|u - u_0(s)| \leq \delta$ with some $\delta > 0$. Outside this set, the kernel $K(t, s, u)$ is involved neither in (2.1) nor in the corresponding collocation method described in Section 2.2. Nevertheless, we remain with the formulation of (V1) given above.

It follows from (V1) that the kernel $K(t, s, u)$ may possess a weak singularity as $s \rightarrow t$ ($i = j = k = 0$, $0 \leq \nu < 1$). In the case $\nu < 0$, the kernel $K(t, s, u)$ is bounded for $0 \leq s \leq t \leq T$ and fixed $u \in \mathbb{R}$, but its derivatives may be singular as $s \rightarrow t$. Often the kernel K has the form

$$K(t, s, u) = a(t, s, u)(t - s)^{-\beta}, \quad \beta < 1,$$

or

$$K(t, s, u) = a(t, s, u) \log(t - s),$$

where $a(t, s, u)$ is an m -smooth function of its arguments for $0 \leq s \leq t \leq T$, $u \in \mathbb{R}$. Clearly, condition (V1) is satisfied in these examples with $\nu = \beta$ and $\nu = 0$, respectively. Actually, (V1) remains fulfilled even if derivatives of $a(t, s, u)$ have certain (sufficiently weak) singularities as $s \rightarrow t$.

The following result states the regularity properties of solutions of equation (2.1).

Theorem 2.1. *Let the conditions (V1) and (V2) be fulfilled. If equation (2.1) has a solution u in $L^\infty(0, T)$, then $u \in C^{m,\nu}(0, T)$.*

The proof of Theorem 2.1 is given in Section 4. We remark that smoothness properties of solutions to (more special) weakly singular Volterra equations are analyzed in [19, p. 6].

We note also that $u \in C^{m,\nu}(0, T]$ having an integrable derivative in $(0, T]$ can be extended up to a continuous function on $[0, T]$. The extended function will be denoted again by u .

2.2. Piecewise polynomial approximation of the solution. For given $N \in \mathbb{N}$ let $0 = t_0 < t_1 < \dots < t_N = T$ be a partition of the interval $[0, T]$ with grid points

$$(2.5) \quad t_j \equiv t_j^{(N)} = (j/N)^r T, \quad j = 0, 1, \dots, N,$$

where $r \in \mathbb{R}$, $r \geq 1$. For $r = 1$, the grid points t_0, t_1, \dots, t_N are distributed uniformly; for $r > 1$, the grid points are more densely clustered near the left endpoint of the interval $[0, T]$. In every subinterval $[t_{j-1}, t_j]$ ($j = 1, \dots, N$) we choose m collocation points

$$(2.6) \quad \xi_{ji} = t_j + \frac{\eta_i + 1}{2}(t_{j+1} - t_j), \quad i = 1, \dots, m,$$

where η_1, \dots, η_m do not depend on N and satisfy

$$(2.7) \quad -1 \leq \eta_1 < \eta_2 < \dots < \eta_m \leq 1.$$

To a continuous function $u: [0, T] \rightarrow \mathbb{R}$ we assign a piecewise polynomial interpolant $P_N u: [0, T] \rightarrow \mathbb{R}$ as follows: 1) on every subinterval $[t_{j-1}, t_j]$ ($j = 1, \dots, N$), $P_N u$ is a polynomial of degree $m - 1$; 2) $P_N u$ interpolates u at the points $\xi_{j1}, \dots, \xi_{jm}$:

$$(P_N u)(\xi_{ji}) = u(\xi_{ji}), \quad i = 1, \dots, m; \quad j = 1, \dots, N.$$

Thus, the interpolation function $(P_N u)(t)$ is independently defined in every subinterval $[t_{j-1}, t_j]$ ($j = 1, \dots, N$) and may be discontinuous at the interior grid points $t = t_j$ ($j = 1, \dots, N - 1$). We may treat $P_N u$ as a two-valued function at these points. Note that in the case $\eta_1 = -1$ and $\eta_m = 1$, $P_N u$ is a continuous function on $[0, T]$.

Let E_N denote the range of the interpolatory projection P_N , that is, the set of all piecewise polynomial functions on $[0, T]$ which are polynomials of degree not exceeding $m - 1$ on every subinterval $[t_{j-1}, t_j]$ ($j = 1, \dots, N$). We introduce also the notation

$$h = T/N.$$

We look for an approximate solution $u_N \in E_N$ to the integral equation (2.1) which satisfies this equation at the collocation points (2.6):

$$(2.8) \quad [u_N(t) - \int_0^t K(t, s, u_N(s)) ds - f(t)]_{t=\xi_{ji}} = 0, \quad i = 1, \dots, m; \quad j = 1, \dots, N.$$

Conditions (2.8) form a system of equations whose exact form is determined by the choice of a basis of E_N . For instance, in the interval $[t_{j-1}, t_j]$ ($j = 1, \dots, N$)

we may use the representation

$$(2.9) \quad u_N(s) = \sum_{i=1}^m c_{ji} \varphi_{ji}(s), \quad s \in [t_{j-1}, t_j],$$

where $\varphi_{ji}(s)$ ($s \in [t_{j-1}, t_j]$) is the polynomial of degree $m - 1$ such that

$$\varphi_{ji}(\xi_{jk}) = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{if } k \neq i, \end{cases} \quad k = 1, \dots, m.$$

The collocation conditions (2.8) then lead to the following (nonlinear) system of algebraic equations for the coefficients $\{c_{ji}\}$:

$$(2.10) \quad c_{ji} = f(\xi_{ji}) + \sum_{l=1}^{j-1} \int_{t_{l-1}}^{t_l} K(\xi_{ji}, s, \sum_{k=1}^m c_{lk} \varphi_{lk}(s)) ds + \int_{t_{j-1}}^{\xi_{ji}} K(\xi_{ji}, s, \sum_{k=1}^m c_{jk} \varphi_{jk}(s)) ds, \quad i = 1, \dots, m; j = 1, \dots, N.$$

This represents a recursive process: First, the coefficients c_{11}, \dots, c_{1m} can be found by solving the system

$$(2.11) \quad c_{1i} = f(\xi_{1i}) + \int_0^{\xi_{1i}} K(\xi_{1i}, s, \sum_{k=1}^m c_{1k} \varphi_{1k}(s)) ds, \quad i = 1, \dots, m.$$

Using c_{11}, \dots, c_{1m} , one can then find c_{21}, \dots, c_{2m} from the system

$$c_{2i} = f(\xi_{2i}) + \int_0^{t_1} K(\xi_{2i}, s, \sum_{k=1}^m c_{1k} \varphi_{1k}(s)) ds + \int_{t_1}^{\xi_{2i}} K(\xi_{2i}, s, \sum_{k=1}^m c_{2k} \varphi_{2k}(s)) ds, \quad i = 1, \dots, m.$$

Generally, using $c_{11}, \dots, c_{1m}, \dots, c_{j-1,1}, \dots, c_{j-1,m}$, the coefficients c_{j1}, \dots, c_{jm} can be found from the m equations (2.10) with corresponding j . Thus at every step (on every subinterval $[t_{j-1}, t_j]$) one has to solve a system of m nonlinear equations; the initial guess can be chosen by using the solution on the previous subinterval. For the first block system (2.11) a suitable initial guess is $c_{1i} = f(\xi_{1i}), i = 1, \dots, m$.

The following theorem states the global convergence rate for the collocation method (2.8).

Theorem 2.2. *Let assumptions (V1)–(V3) be fulfilled and let the collocation points (2.6) corresponding to the gridpoints (2.5) be used.*

Then there exist an $N_0 \in \mathbb{N}$ and a real number $\delta_0 > 0$ such that, for $N > N_0$, the collocation method (2.8) defines a unique approximation $u_N \in E_N$ to the solution u_0 of equation (2.1) satisfying $\|u_N - u_0\|_{L^\infty(0,T)} \leq \delta_0$. The following error estimates hold:

1) *if $m < 1 - \nu$ then*

$$(2.12) \quad \max_{0 \leq t \leq T} |u_N(t) - u_0(t)| \leq ch^m \quad \text{for } r \geq 1;$$

2) if $m = 1 - \nu$ then

$$(2.13) \quad \max_{0 \leq t \leq T} |u_N(t) - u_0(t)| \leq c \begin{cases} h^m(1 + |\log h|) & \text{for } r = 1, \\ h^m & \text{for } r > 1, \end{cases}$$

and

$$(2.14) \quad \|u_N - u_0\|_{L^p(0,T)} \leq ch^m \quad \text{for } r \geq 1, 1 \leq p < \infty;$$

3) if $m > 1 - \nu$ then

$$(2.15) \quad \max_{0 \leq t \leq T} |u_N(t) - u_0(t)| \leq c \begin{cases} h^{r(1-\nu)} & \text{for } 1 \leq r \leq \frac{m}{1-\nu}, \\ h^m & \text{for } r \geq \frac{m}{1-\nu}. \end{cases}$$

and for $1 \leq p < \infty$,

$$(2.16) \quad \|u_N - u_0\|_{L^p(0,T)} \leq c \begin{cases} h^{r(1-\nu+\frac{1}{p})} & \text{if } 1 \leq r < \frac{m}{1-\nu+\frac{1}{p}}, m > 1 - \nu + \frac{1}{p}, \\ h^m(1 + |\log h|)^{\frac{1}{p}} & \text{if } r = \frac{m}{1-\nu+\frac{1}{p}}, m \geq 1 - \nu + \frac{1}{p}, \\ h^m & \text{if } r > \frac{m}{1-\nu+\frac{1}{p}}, r \geq 1; \end{cases}$$

4) if $r = r(m, \nu) \geq 1$ is restricted by conditions

$$(2.17) \quad \begin{aligned} r &> \frac{m}{2(1-\nu)} && \text{for } 0 \leq \nu < 1, \\ r &> \frac{m}{2-\nu} && \text{for } \nu < 0, \\ r &\geq 1 && \text{for } \nu < 0, \nu < -(m-2), \end{aligned}$$

then

$$(2.18) \quad \varepsilon_N \leq ch^m,$$

where

$$(2.19) \quad \varepsilon_N = \max_{i=1, \dots, m; j=1, \dots, N} |u_N(\xi_{ji}) - u_0(\xi_{ji})|$$

is the maximal error of the approximate solution $u_N \in E_N$ at the collocation points (2.6). The constants c in (2.12)–(2.18) are independent of $h = T/N$.

The proof of Theorem 2.2 is given in Section 4.

2.3. Superconvergence at the collocation points. We now assume that the points η_1, \dots, η_m in (2.6) (the points (2.7)) are the knots of the quadrature formula

$$(2.20) \quad \int_{-1}^1 \varphi(\xi) d\xi \approx \sum_{i=1}^m w_i \varphi(\eta_i), \quad -1 \leq \eta_1 < \dots < \eta_m \leq 1,$$

which is exact for polynomials of degree $m + \mu$, $\mu \in \mathbf{Z}$, $0 \leq \mu \leq m - 1$. Actually, the weights w_1, \dots, w_m of the quadrature formula (2.20) will not be used in our analysis. The case $\mu = m - 1$ corresponds to the Gauss quadrature formula and is of the greatest interest in the following theorem.

Theorem 2.3. *Let the following conditions be fulfilled:*

(V1') *The kernel $K(t, s, u)$ and $\partial K(t, s, u)/\partial u$ are $m + \mu + 1$ times ($m, \mu \in \mathbf{Z}$, $m \geq 1$, $0 \leq \mu \leq m - 1$) continuously differentiable with respect to t, s, u for $t \in [0, T]$, $s \in [0, t]$, $u \in \mathbb{R}$, and satisfy (2.2) and (2.3) with $i + j + k \leq m + \mu + 1$, $\nu \in (-\infty, 1)$.*

- (V2') $f \in C^{m+\mu+1,\nu}(0, T]$.
- (V3) The integral equation (2.1) has a solution $u_0 \in L^\infty(0, T)$.
- (V4) The collocation points (2.6) are generated by the knots (2.7) of a quadrature formula (2.20) which is exact for all polynomials of degree $m + \mu$, $0 \leq \mu \leq m - 1$.
- (V5) The scaling parameter $r = r(m, \nu, \mu) \geq 1$ is subject to the restrictions

$$(2.21) \quad \begin{aligned} r &> \frac{m}{1-\nu}, \quad r \geq \frac{m+1-\nu}{2-\nu} && \text{if } 1-\nu < \mu+1, \\ r &> \frac{m}{1-\nu}, \quad r > \frac{m+\mu+1}{2-\nu} && \text{if } \mu+1 \leq 1-\nu < m, \\ r &\geq \frac{m+\mu+1}{2-\nu}, \quad r > 1 && \text{if } 1-\nu = m, \\ r &\geq \frac{m+\mu+1}{2-\nu} && \text{if } 1-\nu > m. \end{aligned}$$

Then

$$(2.22) \quad \varepsilon_N \leq ch^m \begin{cases} h & \text{if } \nu < 0, \\ h(1 + |\log h|) & \text{if } \nu = 0, \\ h^{1-\nu} & \text{if } \nu > 0, \end{cases}$$

where ε_N is defined in (2.19) and c is a positive constant which is independent of $h = T/N$.

The proof of Theorem 2.3 is given in Section 4.

Theorem 2.4. *Let the conditions of Theorem 2.3 hold. Assume additionally that:*

- (V6) $\nu < 0$, $\mu \geq 1$, and for $0 \leq j \leq \min\{\mu - 1, -\nu\}$, $0 \leq k \leq \min\{\mu - 1, -\nu\}$, the derivatives

$$(2.23) \quad \left(\frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^{k+1} K(t, s, u)$$

are bounded and continuous for $0 \leq t \leq T$, $0 \leq s \leq t$, $|u| \leq \tau$ with any $\tau > 0$, and $\left(\frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^{k+1} K(t, s, u) \rightarrow 0$ as $s \rightarrow t - 0$.

Then

$$(2.24) \quad \varepsilon_N \leq ch^m \begin{cases} h^{\mu+1} & \text{if } 1-\nu > \mu+1, \\ h^{\mu+1}(1 + |\log h|) & \text{if } 1-\nu = \mu+1, \\ h^{1-\nu} & \text{if } 1-\nu < \mu+1, \end{cases}$$

where ε_N is defined in (2.19) and the constant c is independent of $h = T/N$.

The proof of Theorem 2.4 is given in Section 4.

Remark 2.1 (comment on the additional condition (V6)). Assumption (V1') of Theorem 2.3 guarantees the boundedness and continuity of the derivatives (2.23) for $j < \min\{\mu - 1, -\nu\}$, $j + k \leq m + \mu + 1$, on any set $0 \leq s < t \leq T$, $-\tau < u < \tau$, $\tau > 0$; for $j = -\nu$ with $\nu \in \mathbf{Z}$, $-\nu < \mu - 1$, a logarithmic singularity may occur. Condition (V6) bans this possible singularity.

Remark 2.2. The estimate $\max_{0 \leq t \leq T} |u_N(t) - u_0(t)| \leq ch^m$ is of optimal order even for a function $u \in C^\infty[0, T]$. Theorem 2.2 shows that, using sufficiently great values of the scaling parameters r , the optimal accuracy $O(h^m)$ can be achieved for the collocation method (2.8). Theorems 2.3 and 2.4 show that superconvergence occurs at the collocation points.

Remark 2.3. Under the conditions of Theorem 2.3

$$\max_{0 \leq t \leq T} |\tilde{u}_N(t) - u_0(t)| \leq ch^m \begin{cases} h & \text{if } \nu < 0, \\ h(1 + |\log h|) & \text{if } \nu = 0, \\ h^{1-\nu} & \text{if } \nu > 0, \end{cases}$$

where

$$\tilde{u}_N(t) = \int_0^t K(t, s, u_N(s))ds + f(t).$$

Under the conditions of Theorem 2.4,

$$\max_{0 \leq t \leq T} |\hat{u}_N(t) - u_0(t)| \leq ch^m \begin{cases} h^{\mu+1} & \text{if } 1 - \nu > \mu + 1, \\ h^{\mu+1}(1 + |\log h|) & \text{if } 1 - \nu = \mu + 1, \\ h^{1-\nu} & \text{if } 1 - \nu < \mu + 1. \end{cases}$$

3. FREDHOLM EQUATIONS WITH WEAKLY SINGULAR KERNELS

In this section we formulate some results from [39, pp. 22–23] about Fredholm integral equations which give a basis for the proofs of the results presented in Section 2.

Consider the nonlinear Fredholm integral equation

$$(3.1) \quad u(t) = \int_0^{2T} K(t, s, u(s))ds + f(t), \quad 0 \leq t \leq 2T.$$

We make the following assumptions (F1) and (F2).

(F1) The kernel $K = K(t, s, u)$ is m times ($m \geq 1$) continuously differentiable with respect to t, s, u for $t, s \in [0, 2T]$, $t \neq s$, $u \in \mathbb{R}$, and there exists a real number $\nu \in (-\infty, 1)$ such that for nonnegative integers i, j, k with $i + j + k \leq m$ inequalities (2.2) and (2.3) hold.

(F2) $f \in C^{m,\nu}(0, 2T)$, i.e. $f(t)$ is m times continuously differentiable for $0 < t < 2T$, and the estimate

$$(3.2) \quad |f^{(k)}(t)| \leq \text{const}_f \begin{cases} 1 & \text{if } k < 1 - \nu, \\ 1 + |\log \varrho(t)| & \text{if } k = 1 - \nu, \\ \varrho(t)^{1-\nu-k} & \text{if } k > 1 - \nu, \end{cases} \quad t \in (0, 2T),$$

holds for $k = 0, 1, \dots, m$, where $\varrho(t) = \min\{t, 2T - t\}$.

Theorem 3.1 ([23], [39, p. 137]). *Let conditions (F1) and (F2) be fulfilled. If the integral equation (3.1) has a solution $u \in L^\infty(0, 2T)$, then $u \in C^{m,\nu}(0, 2T)$.*

In the sequel we shall assume that:

(F3) The integral equation (3.1) has a solution $u_0 \in L^\infty(0, 2T)$ and the linearized equation

$$(3.3) \quad v(t) = \int_0^{2T} K_0(t, s)v(s)ds, \quad K_0(t, s) = \partial K(t, s, u)/\partial u \Big|_{u=u_0(s)},$$

has in $L^\infty(0, 2T)$ only the trivial solution $v \equiv 0$.

For given $N \in \mathbb{N}$, let $0 = t_0 < t_1 < \dots < t_{2N} = 2T$ be a partition of the interval $[0, 2T]$ with gridpoints (2.5) and gridpoints t_{N+1}, \dots, t_{2N} which are obtained by reflecting t_{N-1}, \dots, t_0 with respect to $t_N = T$:

$$(3.4) \quad t_j = (j/N)^r T, \quad j = 0, 1, \dots, N; \quad t_{j+N} = 2T - t_{N-j}, \quad j = 1, \dots, N.$$

In every subinterval $[t_{j-1}, t_j]$ ($j = 1, \dots, 2N$) we choose m collocation points (2.6) determined by the points (2.7). We denote by \tilde{E}_N the set of all piecewise polynomial functions on $[0, 2T]$ which are polynomials of degree not exceeding $m - 1$ on every interval $[t_{j-1}, t_j]$ ($j = 1, \dots, 2N$). We look for an approximate solution $u_N \in \tilde{E}_N$ to the integral equation (3.1) which satisfies this equation at the collocation points (2.6) corresponding to the gridpoints (3.4):

$$(3.5) \quad \left[u_N(t) - \int_0^{2T} K(t, s, u_N(s)) ds - f(t) \right]_{t=\xi_{ji}} = 0,$$

$$i = 1, \dots, m; \quad j = 1, \dots, 2N.$$

Theorem 3.2 (see [39, p. 143], cf. also [43], [42]). *Let conditions (F1)–(F3) be fulfilled and let the collocation points (2.6) with gridpoints given by (3.4) be used.*

Then there exist an $N_0 \in \mathbb{N}$ and a real number $\delta_0 > 0$ such that for $N > N_0$, the collocation method (3.5) defines a unique approximation $u_N \in \tilde{E}_N$ to the solution u_0 of equation (3.1) satisfying $\|u_N - u_0\|_{L^\infty(0,2T)} \leq \delta_0$. The following error estimates hold:

- 1) if $m < 1 - \nu$ then
- $$(3.6) \quad \max_{0 \leq t \leq 2T} |u_N(t) - u_0(t)| \leq ch^m \quad \text{for } r \geq 1;$$
- 2) if $m = 1 - \nu$ then
- $$(3.7) \quad \max_{0 \leq t \leq 2T} |u_N(t) - u_0(t)| \leq c \begin{cases} h^m(1 + \log h) & \text{for } r = 1, \\ h^m & \text{for } r > 1, \end{cases}$$

and

$$(3.8) \quad \|u_N - u_0\|_{L^p(0,2T)} \leq ch^m \quad \text{for } r \geq 1, \quad 1 \leq p < \infty;$$

- 3) if $m > 1 - \nu$ then
- $$(3.9) \quad \max_{0 \leq t \leq 2T} |u_N(t) - u_0(t)| \leq c \begin{cases} h^{r(1-\nu)} & \text{for } 1 \leq r \leq \frac{m}{1-\nu}, \\ h^m & \text{for } r \geq \frac{m}{1-\nu}, \end{cases}$$

and

$$(3.10) \quad \|u_N - u_0\|_{L^p(0,2T)} \leq c \begin{cases} h^{r(1-\nu+\frac{1}{p})} & \text{if } 1 \leq r < \frac{m}{1-\nu+\frac{1}{p}}, \quad m > 1 - \nu + \frac{1}{p}, \\ h^m(1 + |\log h|)^{\frac{1}{p}} & \text{if } r = \frac{m}{1-\nu+\frac{1}{p}}, \quad m \geq 1 - \nu + \frac{1}{p}, \\ h^m & \text{if } r > \frac{m}{1-\nu+\frac{1}{p}}, \quad r \geq 1, \end{cases}$$

for $1 \leq p < \infty$;

4) if $r = r(m, \nu) \geq 1$ satisfies (2.4), then

$$(3.11) \quad \tilde{\varepsilon}_N \leq ch^m$$

where

$$(3.12) \quad \tilde{\varepsilon}_N = \max_{i=1, \dots, m; j=1, \dots, 2N} |u_N(\xi_{ji}) - u_0(\xi_{ji})|.$$

Theorem 3.3 ([24]). *Let the following conditions be fulfilled:*

- (F1') *The kernel $K(t, s, u)$ and $\partial K(t, s, u)/\partial u$ are $m + \mu + 1$ times ($m, \mu \in \mathbf{Z}$, $m \geq 1$, $0 \leq \mu \leq m - 1$) continuously differentiable with respect to t, s, u for $t, s \in [0, 2T]$, $t \neq s$, $u \in \mathbb{R}$, and satisfy (2.2) and (2.3) with $i + j + k \leq m + \mu + 1$, $-\infty < \nu < 1$.*
- (F2') *$f \in C^{m+\mu+1, \nu}(0, 2T)$.*
- (F3) *The integral equation (3.1) has a solution $u_0 \in L^\infty(0, 2T)$, and the linearized equation (3.3) has in $L^\infty(0, 2T)$ only the trivial solution $v \equiv 0$.*
- (F4) *The collocation points (2.6) are generated by the knots (2.7) of a quadrature formula (2.20) which is exact for all polynomials of degree $m + \mu$, $0 \leq \mu \leq m - 1$.*
- (F5) *The scaling parameter $r = r(m, \nu, \mu) \geq 1$ satisfies the conditions (2.18).*

Then

$$\tilde{\varepsilon}_N \leq ch^m \begin{cases} h & \text{if } \nu < 0, \\ h(1 + |\log h|) & \text{if } \nu = 0, \\ h^{1-\nu} & \text{if } \nu > 0, \end{cases}$$

where $\tilde{\varepsilon}_N$ is defined in (3.12).

If, in addition, we assume

- (F6) $\nu < 0$, $\mu \geq 1$, and for $0 \leq j \leq \min\{\mu - 1, -\nu\}$, $0 \leq k \leq \min\{\mu - 1, -\nu\}$, the derivatives (2.23) are bounded and continuous on $[0, 2T] \times [0, 2T] \times [-\tau, \tau]$ with any $\tau > 0$,

then

$$\tilde{\varepsilon}_N \leq ch^m \begin{cases} h^{\mu+1} & \text{if } 1 - \nu > \mu + 1, \\ h^{\mu+1}(1 + |\log h|) & \text{if } 1 - \nu = \mu + 1, \\ h^{1-\nu} & \text{if } 1 - \nu < \mu + 1. \end{cases}$$

4. PROOF OF THEOREMS 2.1–2.4

Assume that the kernel $K(t, s, u)$ of equation (2.1) satisfies (V1) for $0 \leq s < t \leq T$, $-\infty < u < \infty$. Assume also that the forcing function $f(t)$ of equation (2.1) satisfies (V2) for $0 < t \leq T$, and let $u_0 \in L^\infty(0, T)$ be a solution of equation (2.1). First of all we extend $K(t, s, u)$ up to a $\tilde{K}(t, s, u)$ which will satisfy condition (F1) for $0 \leq t \leq 2T$, $0 \leq s \leq 2T$, $t \neq s, u \in \mathbb{R}$.

We shall use an extension method based on reflecting (see, for example [17]): for a given function $v \in C^m[a - \tau, a]$, $m \in \mathbb{N}$, $a, \tau \in \mathbb{R}$, $\tau > 0$, put

$$(4.1) \quad \tilde{v}(s) = \begin{cases} v(s) & \text{for } s \in [a - \tau, a], \\ \sum_{j=0}^m c_j v(a - j(s - a)) & \text{for } s \in (a, a + \frac{\tau}{m}], \end{cases}$$

where

$$(4.2) \quad \sum_{j=0}^m (-j)^k c_j = 1, \quad k = 0, 1, \dots, m.$$

From (4.1) and (4.2) one obtains that

$$\tilde{v} \in C^m \left[a - \tau, a + \frac{\tau}{m} \right].$$

We now construct an extension for the kernel $K(t, s, u)$ of equation (1.1). First, for any $u \in \mathbb{R}$, $\frac{2}{3}T \leq t \leq T$, we extend $K(t, s, u)$ with respect to s for $s < 0$ (denoting this extension again by $K(t, s, u)$):

$$K(t, s, u) = e(-s) \sum_{j=0}^m c_j K(t, -js, u), \quad -\frac{2T}{3m} \leq s < 0.$$

Here (c_0, c_1, \dots, c_m) is the (unique) solution of the system (4.2) and $e \in C^\infty(\mathbb{R}_+)$ is a smooth function such that

$$(4.3) \quad e(s) = \begin{cases} 1 & \text{for } 0 \leq s \leq \frac{2T}{9m}, \\ 0 & \text{for } \frac{2T}{6m} \leq s \leq T. \end{cases}$$

Second, for any $u \in \mathbb{R}$, we extend $K(t, s, u)$ with respect to t ($T \leq t \leq 2T$) along the lines $s = t - \gamma$ ($0 < \gamma \leq 2T$):

$$K_1(t, s, u) = \begin{cases} K(t, s, u) & \text{for } 0 \leq s < t \leq T, \\ e(t-T) \sum_{j=0}^m c_j K(T-j(t-T), T-j(s+\gamma-T) - \gamma, u) & \text{for } T \leq s < t \leq 2T, \quad t-s = \gamma. \end{cases}$$

Finally, in the third step we put

$$(4.4) \quad \tilde{K}(t, s, u) = \begin{cases} K_1(t, s, u) e_1(u) & \text{for } 0 \leq s < t \leq 2T, \\ 0 & \text{for } 0 \leq t < s \leq 2T, \end{cases}$$

with an $e_1 \in C_0^\infty(\mathbb{R})$, $e_1(u) = 1$ for $|u| \leq 2M$, $e_1(u) = 0$ for $|u| \geq 3M$, where $M = \max_{0 \leq t \leq T} |u_0(t)|$. It is easy to check that the function $\tilde{K}(t, s, u)$ satisfies (F1).

Indeed, using (V1) we obtain that $\tilde{K}(t, s, u)$ is m times continuously differentiable with respect to t, s, u for $t \in [0, 2T]$, $s \in [0, 2T]$, $t \neq s$, $u \in \mathbb{R}$. Furthermore, it follows from the above construction that:

1) if $0 \leq s < t \leq T$, $u \in \mathbb{R}$, then

$$\begin{aligned} & \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j \left(\frac{\partial}{\partial u} \right)^k \tilde{K}(t, s, u) \\ &= \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j \left(\frac{\partial}{\partial u} \right)^k [K(t, s, u) e_1(u)]; \end{aligned}$$

2) if $T \leq s < t \leq 2T$, $u \in \mathbb{R}$, then

(4.5)

$$\begin{aligned} & \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k \tilde{K}(t, s, u) \\ &= \left[\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k e^{(t-T)} \right] \\ & \quad \times \sum_{p=0}^m c_p K(T-p(t), T-p(s+\gamma-T)-\gamma, u) e_1(u) \\ & \quad + e^{(t-T)} \sum_{p=0}^m c_p \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k \\ & \quad \times [K(T-p(t-T), T-p(s+\gamma-T)-\gamma, u) e_1(u)]; \end{aligned}$$

3) if $0 \leq t < s \leq 2T$ then

$$\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k \tilde{K}(t, s, u) = 0.$$

In case 1) we obtain the estimates (2.2) and (2.3) for $\tilde{K}(t, s, u)$ using Assumption (V1). In case 3) statements (2.2) and (2.3) with respect to $\tilde{K}(t, s, u)$ are trivially fulfilled.

Now consider case 2). Using (4.3), (4.5) and (V1), we are led to

$$\begin{aligned} & \left| \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k \tilde{K}(t, s, u) \right| \\ & \leq cb_1(3M) \left[\begin{array}{ll} 1 & \text{if } \nu < 0 \\ 1 + |\log|t-s|| & \text{if } \nu = 0 \\ |t-s|^{-\nu} & \text{if } \nu > 0 \end{array} \right] + \left[\begin{array}{ll} 1 & \text{if } \nu + i < 0 \\ 1 + |\log|t-s|| & \text{if } \nu + i = 0 \\ |t-s|^{-\nu-i} & \text{if } \nu + i > 0 \end{array} \right] \\ & \leq c'b_1(3M) \begin{cases} 1 & \text{if } \nu + i < 0, \\ 1 + |\log|t-s|| & \text{if } \nu + i = 0, \\ |t-s|^{-\nu-i} & \text{if } \nu + i > 0. \end{cases} \end{aligned}$$

In a similar way we verify that condition (2.3) is also satisfied.

So, we have shown that the function $\tilde{K}(t, s, u)$ is m times continuously differentiable with respect to t, s, u for $t, s, u \in [0, 2T]$, $t \neq s$, $u \in \mathbb{R}$, and satisfies conditions (2.2) and (2.3) with some b_1 and b_2 independent of $|u|$ and $\max\{|u_1|, |u_2|\}$, respectively. Thus, $\tilde{K}(t, s, u)$ also satisfies the global Lipschitz condition: for $t \in [0, 2T]$, $s \in [0, t)$, $u_1, u_2 \in \mathbb{R}$,

$$(4.6) \quad |\tilde{K}(t, s, u_1) - \tilde{K}(t, s, u_2)| \leq \tilde{b} \begin{cases} 1 & \text{if } \nu < 0, \\ 1 + |\log|t-s|| & \text{if } \nu = 0, \\ |t-s|^{-\nu} & \text{if } \nu > 0, \end{cases}$$

with a constant \tilde{b} not depending on u_1 and u_2 .

We now define

$$(4.7) \quad \tilde{f}(t) = \begin{cases} f(t) & \text{for } 0 < t \leq T, \\ e(t-T) \sum_{j=0}^m c_j f(T-j(t-T)) & \text{for } T < t \leq 2T, \end{cases}$$

where (c_0, c_1, \dots, c_m) is the solution of the system (4.2) and $e \in C^\infty(\mathbb{R}_+)$ is defined in (4.3). It is clear that $\tilde{f} \in C^{m,\nu}(0, 2T) \subset C[0, 2T]$.

Consider the equation

$$(4.8) \quad u(t) = \int_0^{2T} \tilde{K}(t, s, u(s)) ds + \tilde{f}(t), \quad 0 \leq t \leq 2T,$$

where \tilde{K} and \tilde{f} are defined by (4.4) and (4.7). Equation (4.8) is actually a Volterra integral equation, since $\tilde{K}(t, s, u) = 0$ for $0 \leq t < s \leq 2T$. Due to the weak singularity of the kernel $\tilde{K}(t, s, u)$ and the global Lipschitz condition (4.6), (4.8) is uniquely solvable in $C[0, 2T]$; let $\tilde{u}_0 \in C[0, 2T]$ denote the solution. It follows from the above construction that $\tilde{u}_0(t) = u_0(t)$ for $0 \leq t \leq T$. According to Theorem 3.1, $\tilde{u}_0 \in C^{m,\nu}(0, 2T)$; therefore $u_0 \in C^{m,\nu}(0, T]$. Theorem 2.1 is thus proved.

Further, for $0 \leq t \leq 2T$, the linear homogenous Fredholm equation

$$v(t) = \int_0^{2T} \tilde{K}_0(t, s)v(s) ds, \quad \tilde{K}_0(t, s) = \left. \frac{\partial \tilde{K}(t, s, u)}{\partial u} \right|_{u=\tilde{u}_0(s)},$$

actually takes the form of a linear homogeneous Volterra equation,

$$v(t) = \int_0^t \tilde{K}_0(t, s)v(s) ds,$$

and therefore has in $L^\infty(0, 2T)$ only the trivial solution $v \equiv 0$. Thus condition (F3) is fulfilled with respect to the equation (4.8). Finally, assume that the gridpoints (3.4) are used. Then the assertions of Theorem 2.2 follow immediately from the corresponding statements of Theorem 3.2 about equation (4.8).

Thus, Theorem 2.2 is proved.

In a similar way we obtain that the function $\tilde{K}(t, s, u)$ in (4.4) satisfies the conditions (F1') and (F6) whenever the kernel $K(t, s, u)$ of the equation (2.1) satisfies the conditions (V1') and (V6), respectively. Finally we obtain that the function $\tilde{f}(t)$ in (4.6) satisfies (F2') as long as the forcing function $f(t)$ obeys (V2'), and condition (F3) will be fulfilled with respect to (4.8) as long as equation (2.1) is solvable in $L^\infty(0, T)$. Therefore, assuming that the conditions of Theorem 2.3 and Theorem 2.4, respectively, are fulfilled, the statements of Theorems 2.3 and 2.4 follow immediately from Theorem 3.3.

Hence, Theorems 2.3 and 2.4 are proved.

5. NUMERICAL EXAMPLE

Let $m = 2$, and let $\eta_1 = -\frac{1}{\sqrt{3}}$ and $\eta_2 = \frac{1}{\sqrt{3}}$ be the knots of the Gauss quadrature formula $\int_{-1}^1 \varphi(\xi) d\xi \approx \varphi(\eta_1) + \varphi(\eta_2)$. In this case the approximate solution $u_N \in E_N$ to equation (2.1) in the interval $[t_{j-1}, t_j]$ ($j = 1, \dots, N$) can be represented in the form

$$(5.1) \quad u_N(t) = c_{j1} \frac{\xi_{j2} - t}{\xi_{j2} - \xi_{j1}} + c_{j2} \frac{t - \xi_{j1}}{\xi_{j2} - \xi_{j1}}, \quad t_{j-1} \leq t \leq t_j,$$

where ξ_{j1}, ξ_{j2} are the knots (2.6) and the coefficients $c_{jk} = u_N(\xi_{jk})$ ($k = 1, 2; j = 1, \dots, N$) are determined from the system (2.10) ($m = 2$). From Theorem 2.2 we obtain for the approximate solution (5.1) the estimate

$$(5.2) \quad \varepsilon_N \leq ch^2 \quad \text{for} \quad \begin{cases} r > \frac{1}{1-\nu} & \text{if } \nu \geq 0, \\ r \geq 1 & \text{if } \nu < 0, \end{cases}$$

where $h = T/N$ and

$$(5.3) \quad \varepsilon_N = \max_{i=1,2;j=1,\dots,N} |u_N(\xi_{ji}) - u_0(\xi_{ji})|.$$

From Theorems 2.3 and 2.4, we obtain the following estimates for the error (5.3) (see (2.22), (2.24), $m = 2$ with $\mu = 1$):

$$(5.4) \quad \varepsilon_N \leq ch^2 \begin{cases} h & \text{if } \nu < 0, \\ h(1 + |\log h|) & \text{if } \nu = 0, \\ h^{1-\nu} & \text{if } \nu > 0, \end{cases}$$

or (under the conditions of Theorem 2.4)

$$(5.5) \quad \varepsilon \leq ch^2 \begin{cases} h^2 & \text{if } \nu < -1, \\ h^2(1 + |\log h|) & \text{if } \nu = -1, \\ h^{1-\nu} & \text{if } \nu > -1, \end{cases}$$

provided that

$$(5.6) \quad \begin{cases} r \geq 1 & \text{if } \nu < -2, \\ r > \frac{4}{2-\nu} & \text{if } -2 \leq \nu \leq -1, \\ r \geq \frac{3-\nu}{2-\nu} & \text{if } -1 < \nu < 1 - \sqrt{2}, \\ r > \frac{2}{1-\nu} & \text{if } 1 - \sqrt{2} \leq \nu < 1. \end{cases}$$

Now we present the following example. Consider the integral equation

$$(5.7) \quad u(t) = \int_0^t (t-s)^{-1/2} u^2(s) ds + t^{1/2} (1 - \frac{4}{3}t), \quad 0 \leq t \leq T.$$

It is easy to check that $u_0(t) = t^{1/2}$ is the exact solution to equation (5.7) and Assumptions (V1') and (V2') of Theorem 2.3 are fulfilled with $\nu = 1/2$, $m = 2$, $\mu = 1$.

The equation (5.7) was solved numerically by the collocation method (2.10) ($m = 2$), where the Gauss points $\eta_2 = -\eta_1 = 1/\sqrt{3}$ were used for determining the collocation points (2.6), and we chose $r = 41/10 > 2/(1-\nu)$ (see (5.6)). At every step (on every subinterval $[t_{j-1}, t_j]$) the coefficients $c_{ji} = u_N(\xi_{ji})$ ($i = 1, 2$) were calculated from (2.10) ($m = 2$) by the Newton method. All the integrals which are needed for the construction of the system (2.10) were found analytically. Some of the numerical results for $T = 3/4$ are presented in the following Table 1, where ε_N is defined in (5.3). The experiments were carried out on an IBM 4381 (in double precision).

From Table 1 we can see that the numerical results are consistent with the theoretical estimate, which is $\varepsilon_N = O(h^{5/2})$, $h = 0.75/N$. Notice that the number of collocation points (the number of unknowns) is $2N$.

TABLE 1

N	$\varepsilon_N (r = 41/10)$	$\varepsilon_{N/2}/2^{5/2}$
4	0.55 E - 01	
8	0.80 E - 02	0.97 E - 02
16	0.12 E - 02	0.14 E - 02
32	0.19 E - 03	0.21 E - 03
64	0.32 E - 04	0.33 E - 04
128	0.20 E - 05	0.57 E - 05

In the same example on longer intervals (e.g. $T = 4$), some instability of the approximate solution was observed. The numerical stability of the scheme is worth examining independently.

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